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Multiphase evolution of population and its application to optics and colliding-beam experiments

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Abstract. In this paper we have analysed a multiphase evolution of population growth. Individual birth and immigration are assumed to be the consequence of the evolution of an individual through a sequence of phases whose duration form a family of independent non-negative random variables. The population model is then adapted to describe the evolution of photons in a cavity and, in particular, it is shown that a multiphase immigration model corresponds to the photons resulting from a stream obtained by amplitude mixing of coherent and chaotic beams. The model is also shown to bring out the characteristics of the multiplicity distribution of particles produced in high-energy collisions.

1. Introduction

The object of this paper is to analyse an age-dependent model of population growth with special reference to the population point process models of cavity radiation. The motivation for the analysis is twofold. The first is to bring out the spectral properties of the radiation field generated by the population of photons. The second is to arrive at the scaled distribution of hadron multiplicities in a typical colliding-beam experiment. Since it is generally believed that the negative binomial or branching model provides an adequate description of hadron multiplicity distribution, it is hoped that the analysis of multiphase population evolution will bring out the characteristics of the multiplicity distribution that can be checked directly by further analysis of the data from collider-beam experiments.

The layout of the paper is as follows. In section 2 we analyse a realistic model of population growth in which the birth rates are not constant. More specifically, we assume that the members of the population evolve through a certain number of phases of random duration before birth/fission takes place. In addition, it is also assumed that the death rate is a constant in each of the phases. We superimpose on this process a process of immigration which is not necessarily Markov. Throughout it is assumed that there is a constant rate at which the members of the population emigrate. In section 3 we characterise the point process of emigration from such a population in terms of the product densities. Such an approach is useful since the emigration process can be interpreted to be the detection process that is employed to detect the radiation field. In section 4 we discuss the relevance of this model to the population point process description of cavity radiation and its detection. In section 5 we show how this model can be adapted to describe the multiplicity distribution of particles (hadrons) produced in a typical colliding-beam experiment.

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2. Description of the phase-dependent model of population growth

The stochastic evolution of population has been investigated in the past from many points of view and, in particular, age-dependent population growth has been studied with a view to taking into account the variable nature of birth rate during the life history of an individual member (see, for example, Kendall 1949, Harris 1963, Bartlett 1975). In the Kendall model of population growth it is generally assumed that an individual of age x at time t has a probability $\lambda(x)\,\mathrm{d}t$ of giving birth to another individual in the time interval $(t, t+\mathrm{d}t)$ conditional upon its having survived up to time t. However, if the birth rate $\lambda(x)$ is assumed to be of the form

$$\lambda(x) = \mathrm{e}^{-\lambda x}\frac{(\lambda x)^n}{n!} \tag{2.1}$$

it is possible to interpret the lifespan as the sum of a certain number of exponentially distributed random variables (Srinivasan 1988a). Such an interpretation renders the analysis amenable to easy computation of the different statistical characteristics of the population. We use this approach to model a population subject to birth, death, immigration and emigration. It will turn out that these characteristics are sufficient to model a cavity population of photons. We also adapt this model to arrive at the multiplicity distribution of hadrons in a typical colliding-beam experiment.

We now proceed to describe the evolution of population through multiphases. We assume that each of the individuals of the population immediately on its birth goes through a certain number (say n) of phases before it can give birth to another individual. The duration of the phases are assumed to be independent and identically distributed random variables with a common exponential distribution with parameter λ.

Each individual can give birth to another individual in the nth phase at a rate equal to α per unit time. Each individual, independent of other individuals in the population, has a risk (of death) at a constant rate equal to μ in all the phases except the last where the risk rate is taken to be $\lambda+\mu$. This particular choice of death rate is only to ensure the conservation of probability in the nth phase. It is to be noted that in the original phase model interpreted and used by Srinivasan (1986a, b) there is an additional residuary phase in which no births are possible. The emigration process is at a constant rate η per individual in all the phases. We assume that the marginal point process of the epochs of immigrations form an ordinary renewal process whose interval spans are sums of m positive independent random variables, each with a negative exponential distribution. Thus the immigration can be described with the help of an auxiliary discrete process which is itself a Markov chain over the discrete set of states $\{1, 2, \ldots, m\}$. The Markov chain itself undergoes transitions of the type $i \to i+1$ with rates $\beta_i (i = 1, 2, \ldots, m-1)$. However when the Markov chain is in state m the next transition which occurs at a rate β_m takes the state to any of the states j with probability p_j, the transition itself being marked by the actual materialisation of immigration. Technically speaking, the associated Markov chain is really a semi-Markov process inasmuch as the transition from $m \to m$ is possible with rate $\beta_m p_i (i = 1, 2, \ldots m)$. This will become apparent when we write down the equation governing the probability generating function. We now proceed to analyse the model with special reference to the point process of emigration. For convenience we choose $m = 3$ and introduce the following notation:

$X_i(t)$ is the size of the population in phase i at time t, $i = 1, 2, \ldots, n$;
$X(t)$ is the total size of the population at time t;

$Y(t)$ is the state process of immigration;

$$\pi_{ji}(t) = pr\{Y(t) = i \mid Y(0) = j\} \qquad i, j = 1, 2, 3 \tag{2.2}$$

$$g_i(z, t) = E[z^{X(t)} \mid X(0) = X_i(0) = 1, \beta_j = 0] \qquad i = 1, 2, \ldots, n; j = 1, 2, 3 \tag{2.3}$$

$$G_i(z, t) = E[z^{X(t)} \mid X(0) = 0, \ Y(0) = i] \qquad i = 1, 2, 3 \tag{2.4}$$

where E stands for the mathematical expectation of the quantity within the brackets. To obtain the equations satisfied by the functions $g_i(z, t)(i = 1, 2, \ldots, n)$ we note that the single individual that generates the population is in phase i initially and in the interval $(0, \Delta)$ for $i = 1, 2, \ldots, n-1$

(i) moves to phase $(i+1)$ with probability $\lambda\Delta + o(\Delta)$, or

(ii) dies or emigrates with probability $(\mu + \eta)\Delta + o(\Delta)$, or

(iii) continues to be in phase i with the residual probability $1 - (\lambda + \mu + \eta)\Delta + o(\Delta)$.

Thus we have

$$g_i(z, t) = \lambda\Delta g_{i+1}(z, t-\Delta) + (\mu + \eta)\Delta + [1 - (\lambda + \mu + \eta)\Delta] g_i(z, t-\Delta) + o(\Delta). \tag{2.5}$$

On the other hand, if $i = n$ the individual is in the final phase (residual phase) and over the interval $(0, \Delta)$

(i) gives birth to an individual in phase 1 with probability $\alpha\Delta + o(\Delta)$, or

(ii) dies or emigrates with probability $(\lambda + \mu + \eta)\Delta + o(\Delta)$, or

(iii) continues to be in phase n with the residual probability $1 - (\lambda + \mu + \eta + \alpha)\Delta + o(\Delta)$.

Taking all these possibilities we have

$$\begin{aligned} g_n(z, t) = {} & [1 - (\rho + \alpha)\Delta] g_n(z, t-\Delta) + (\lambda + \mu + \eta)\Delta \\ & + \alpha\Delta g_n(z, t-\Delta) g_1(z, t-\Delta) + o(\Delta). \end{aligned} \tag{2.6}$$

Proceeding to the limit as $\Delta \to 0$ we have

$$\frac{\partial g_i(z, t)}{\partial t} = -\rho g_i(z, t) + \lambda g_{i+1}(z, t) + \mu + \eta \qquad i = 1, 2, \ldots, n-1 \tag{2.7}$$

$$\frac{\partial g_n(z, t)}{\partial t} = -(\rho + \alpha) g_n(z, t) + \alpha g_n(z, t) g_1(z, t) + \lambda + \mu + \eta \tag{2.8}$$

with the initial condition

$$g_i(z, 0) = z \qquad i = 1, 2, \ldots, n \tag{2.9}$$

where $\rho = \lambda + \mu + \eta$.

To obtain the equations satisfied by $G_i(z, t)(i = 1, 2, 3)$ we fix our attention on the time interval $(0, \Delta)$ as before. If the state of immigration is in phase 1 (phase 2), it

(i) moves to phase 2 (phase 3) with probability $\beta_1\Delta + o(\Delta)$ $(\beta_2\Delta + o(\Delta))$, or

(ii) continues to be in phase 1 (phase 2) with the residual probability $1 - \beta_1\Delta + o(\Delta)(1 - \beta_2\Delta + o(\Delta))$.

We have

$$G_i(z, t) = (1 - \beta_i\Delta) G_i(z, t-\Delta) + \beta_i\Delta G_{i+1}(z, t-\Delta) + o(\Delta) \qquad i = 1, 2. \tag{2.10}$$

On the other hand, if the state is in phase 3, immigration materialises at a rate β_3, the state of immigration itself undergoing transition to any of the states 1, 2 and 3 with respective probabilities p_1, p_2 and $p_3 (p_1 + p_2 + p_3 = 1)$. Taking into account that the immigrant is in phase 1 of its evolution and will generate population independent of the state of immigration we obtain

$$G_3(z,t)=(1-\beta_3\Delta)G_3(z,t-\Delta)+\beta_3\Delta(p_1G_1(z,t-\Delta)+p_2G_2(z,t-\Delta)+p_3G_3(z,t-\Delta))g_1(z,t-\Delta)+o(\Delta). \tag{2.11}$$

Now proceeding to the limit as $\Delta\to 0$ we obtain

$$\frac{\partial G_i(z,t)}{\partial t}=-\beta_iG_i(z,t)+\beta_iG_{i+1}(z,t)\qquad i=1,2 \tag{2.12}$$

$$\frac{\partial G_3(z,t)}{\partial t}=-\beta_3G_3(z,t)+(\nu_1G_1(z,t)+\nu_2G_2(z,t)+\nu_3G_3(z,t))g_1(z,t). \tag{2.13}$$

where $\nu_i=\beta_3p_i$, $i=1,2,3$ with the initial condition

$$G_i(z,0)=1\qquad i=1,2,3. \tag{2.14}$$

Although it is difficult to solve for G_i explicitly the moments can be readily generated.

2.1. *Moments of the population*

We introduce the moment functions $a_i(t)$, $b_i(t)$, $A_j(t)$ and $B_j(t)(i=1,2,\ldots,n;\ j=1,2,3)$ by

$$a_i(t)=\left.\frac{\partial g_i}{\partial z}\right|_{z=1} \tag{2.15}$$

$$b_i(t)=\left.\frac{\partial^2 g_i}{\partial z^2}\right|_{z=1} \tag{2.16}$$

$$i=1,2,\ldots,n$$

$$A_j(t)=\left.\frac{\partial G_j}{\partial z}\right|_{z=1} \tag{2.17}$$

$$B_j(t)=\left.\frac{\partial^2 G_j}{\partial z^2}\right|_{z=1} \tag{2.18}$$

$$j=1,2,3$$

where for simplicity of notation we denote $g_i(z,t)$ by $g_i(i=1,2,\ldots,n)$ and $G_j(z,t)$ by $G_j(j=1,2,3)$. We then differentiate both sides of equations (2.12) and (2.13) to obtain, for $j=1,2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}A_j(t)=-\beta_jA_j(t)+\beta_jA_{j+1}(t) \tag{2.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A_3(t)=-(\nu_1+\nu_2)A_3(t)+\nu_1A_1(t)+\nu_2A_2(t)+(\nu_1+\nu_2+\nu_3)a_1(t). \tag{2.20}$$

If we again differentiate equations (2.12) and (2.13) to get, for $j=1,2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}B_j(t)=-\beta_jB_j(t)+\beta_jB_{j+1}(t) \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}B_3(t)=-(\nu_1+\nu_2)B_3(t)+\nu_1B_1(t)+\nu_2B_2(t)+(\nu_1+\nu_2+\nu_3)b_1(t)+2(\nu_1A_1(t)+\nu_2A_2(t)+\nu_3A_3(t))a_1(t) \tag{2.22}$$

with the initial conditions

$$A_j(0) = B_j(0) = 0 \qquad j = 1, 2, 3. \tag{2.23}$$

Solving the above equations (2.19)-(2.22) by the Laplace transform (denoted by *) technique with $\beta_1 = \beta_2 + \nu_1 + \nu_2$ and $\mu + \eta = 2(\beta_2 + \nu_1 + \nu_2)$ we get

$$A_1(t) = \frac{\beta_2\beta_3}{D_1}\left(\tfrac{1}{2} - \tfrac{1}{2}\exp[-2(\beta_2 + \nu_1 + \nu_2)t] - \frac{2(\beta_2 + \nu_1 + \nu_2)}{\Gamma}\exp[-(\beta_2 + \nu_1 + \nu_2)t]\sin\frac{\Gamma}{2}t\right) \tag{2.24}$$

$$A_2(t) = \frac{\beta_2\beta_3}{D_1}\left(\tfrac{1}{2} + \tfrac{1}{2}\exp[-2(\beta_2 + \nu_1 + \nu_2)t] - \exp[-(\beta_2 + \nu_1 + \nu_2)t]\cos\frac{\Gamma}{2}t\right) \tag{2.25}$$

$$A_3(t) = \frac{\beta_3}{D_1}\left(\frac{\beta_2}{2} - \frac{(2\nu_1 + 2\nu_2 + \beta_2)}{2}\exp[-(\beta_2 + \nu_1 + \nu_2)t] + (\nu_1 + \nu_2)\exp[-(\beta_2 + \nu_1 + \nu_2)t]\cos\frac{\Gamma}{2}t + \frac{2\beta_2\nu_1}{\Gamma}\exp[-(\beta_2 + \nu_1 + \nu_2)t]\sin\frac{\Gamma}{2}t\right) \tag{2.26}$$

where $D_1 = (\beta_2 + \nu_1 + \nu_2)^2 + \beta_2\nu_1$ and $\Gamma = 2(\beta_2\nu_1)^{1/2}$ and

$$B_1^*(s) = \frac{\beta_2(\beta_2 + \nu_1 + \nu_2)}{sD(s)}[\beta_3 b_1^*(s) + 2L(s)] \tag{2.27}$$

$$B_2^*(s) = \frac{\beta_2(s + \beta_2 + \nu_1 + \nu_2)}{sD(s)}[\beta_3 b_1^*(s) + 2L(s)] \tag{2.28}$$

$$B_3^*(s) = \frac{(s + \beta_2)(s + \beta_2 + \nu_1 + \nu_2)}{sD(s)}[\beta_3 b_1^*(s) + 2L(s)] \tag{2.29}$$

where

$$L(s) = \nu_1 A_1^*(s + 2\beta_2 + 2\nu_1 + 2\nu_2) + \nu_2 A_2^*(s + 2\beta_2 + 2\nu_1 + 2\nu_2) + \nu_3 A_3^*(s + 2\beta_2 + 2\nu_1 + 2\nu_2) \tag{2.30}$$

$$D(s) = s^2 + 2s(\beta_2 + \nu_1 + \nu_2) + (\beta_2 + \nu_1 + \nu_2)^2 + \beta_2\nu_1$$

$$a_i(t) = \exp[-2(\beta_2 + \nu_1 + \nu_2)t] \qquad i = 1, 2, \ldots, n \tag{2.31}$$

$$b_1^*(0) = \frac{\lambda^n}{2(\beta_2 + \nu_1 + \nu_2)(\rho^n - \lambda^n)}. \tag{2.32}$$

The factorial moments of the equilibrium distribution of the population are easily obtained by the use of the Tauberian theorem given by

$$B_1(\infty) = \lim_{s \to 0} sB_1^*(s) = \frac{\beta_2\beta_3}{2D_1}\left(\frac{\lambda^n}{\rho^n - \lambda^n} + \frac{\nu_1\beta_2 + 3\nu_2\beta_2 + 3\nu_3(3\beta_2 + 2\nu_1 + 2\nu_2)}{2D_2}\right) \tag{2.33}$$

$$A_1(\infty) = \beta_2\beta_3/2D_1 \tag{2.34}$$

where $D_2 = 9(\beta_2 + \nu_1 + \nu_2)^2 + \beta_2\nu_1$.

If we introduce $\mathscr{B}$, the measure of bunching, as

$$\mathscr{B} = B_1(\infty)/[A_1(\infty)]^2 \tag{2.35}$$

and set $n=2$, $\nu_1 = \nu_2 = \lambda$, $\nu_3 = 11\lambda/3$, $\beta_2 = 2\sqrt{2}\lambda/3$ we get

$$\mathscr{B} = 1.8743.$$

The significance of the choice of ν_1, ν_2, ν_3 and β_2 will become apparent presently. In what follows we specifically use the choice $n=2$.

3. Emigration process

We are generally interested in the number of individuals emigrated over an arbitrary interval (t_0, t_0+t) (see Shepherd 1981, Jakeman and Shepherd 1984, Shepherd and Jakeman 1987). There are two ways of dealing with the emigration process. The first consists of dealing with $N(t_0, t)$, the number of individuals emigrated over the time interval (t_0, t_0+t). In our model the process becomes stationary and hence the distributional characteristics of the process $N(t_0, t)$ are independent of t_0. We can proceed in a manner analogous to that of section 2 and obtain the differential equations satisfied by the appropriate generating functions. The moments of $N(t)$ can be obtained by differentiating the resulting set of equations with respect to z at $z=1$ of the corresponding generating functions. The structure of the differential equations is the same as in section 2. There is an alternative line of approach in which we can deal with the point process generated by the epochs of emigration. The emigration process can be characterised in terms of the sequence of product densities. For the model under discussion, these are conditional product densities and are defined by

$$f_1(t) = \lim_{\Delta\to 0} \mathrm{pr}\{N(t+\Delta) - N(t) = 1|_{\substack{\text{population in}\\ \text{equilibrium}\\ \text{initially}}}\}/\Delta \tag{3.1}$$

$$\mathrm{f}_2(\mathrm{t}_1, \mathrm{t}_2) = \lim_{\substack{\Delta_1\to 0\\ \Delta_2\to 0}} \mathrm{pr}\{N(t_1+\Delta_1) - N(t_1) = 1,\ N(t_2+\Delta_2) - N(t_2) = 1|_{\substack{\text{population in}\\ \text{equilibrium}\\ \text{initially}}}\}/\Delta_1\Delta_2 \tag{3.2}$$

with higher-order product densities defined in a similar manner.

From the very construction of the model it is clear that the point process is stationary. Hence we have

$$f_1(t) = \text{a constant} \tag{3.3}$$

$$f_2(t_1, t_2) = \text{a function of } |t_2 - t_1| = h_{sty}(|t_2 - t_1|). \tag{3.4}$$

If we confine our attention to the second-order characteristics, we need only identify the constant on the right-hand side of (3.3) and obtain an explicit expression for the function $h_{sty}(\ )$. To make further progress we introduce the conditional product densities by choosing convenient conditioning and then revert back to the equilibrium condition.

Thus accordingly we define

$$h_1^j(t) = \lim_{\Delta\to 0} \mathrm{pr}\{N(t+\Delta) - N(t) = 1|X_1(0) = X(0) = 1\ \beta_j = 0\}/\Delta \qquad j = 1, 2, 3. \tag{3.5}$$

In addition, the equilibrium first-order function $f_1(\ )$ where

$$f_1(\ ) = \lim_{t\to\infty} h_1(t) \tag{3.6}$$

is given by

$$f_1(\ ) = \eta A_1(\infty) = \eta\beta_2\beta_3/2D_1. \tag{3.7}$$

To obtain $h_1^1(t)$ we use the definition (3.5) directly and relate it to the moment $a_1(t)$. Thus we obtain

$$h_1^1(t) = \eta a_1(t) = \eta \exp[-2(\beta_2 + \nu_1 + \nu_2)t]. \tag{3.8}$$

To obtain $h_{sty}(t)(t>0)$, we note that we have the population maintained in equilibrium at the origin at which point of time one of the individuals has emigrated. Taking into account that the emigrated individual could be in any one of the phases, and any one of the members of the remaining population can generate a population tree between 0 and t, we obtain the contribution as

$$h_{sty}(t)|_{\text{first term}} = \eta^2 B_1(\infty) \exp[-2(\beta_2 + \nu_1 + \nu_2)t]. \tag{3.9}$$

The second term arises from the situation that the emigration that occurs at the epoch t could be due to the population generated by an individual arising from the emigration subsequent to the time origin. Since in any case the emigration at the time origin has to be from a population due to an individual immigrating into the system before the time origin we concentrate our attention at the point at which the immigration effectively takes place. If x is the time coordinate of the epoch we note that the state of immigration is described by the three-state semi-Markov process introduced in section 2. We note that the contribution to $h_{sty}(t)$ is given by

$$h_{sty}(t) = \eta B_1(\infty) h_1'(t) + \frac{\eta^2\beta_1\beta_2}{D_1} \sum_{i,j=1}^{3} \int_{-\infty}^{0} \beta_3 p_j \pi_{ji}(-x) A_i(t) a_1(-x)\, \mathrm{d}x. \tag{3.10}$$

Thus we have

$$\begin{aligned} h_{sty}(t) &= \eta^2 B_1(\infty) \exp[-2(\beta_2 + \nu_1 + \nu_2)t] \\ &\quad + \frac{\eta^2\beta_1\beta_2}{D_1} \sum_{i,j=1}^{3} \nu_j \pi_{ij}^*[2(\beta_2 + \nu_1 + \nu_2)] A_i(t) \end{aligned} \tag{3.11}$$

where $\nu_j = \beta_3 p_j, j = 1, 2, 3$.

The expression for $h_{sty}(t)$ is explicit once we determine π_{ji}^*. Making use of the semi-Markov nature of the process and observing that the sojourn time distributions are exponential in nature we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \pi_{ji}(t) = -\beta_j \pi_{ji}(t) + \beta_j \pi_{j+1\,i}(t) \qquad j = 1, 2 \tag{3.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \pi_{3i}(t) = -(\nu_1 + \nu_2)\pi_{3i}(t) + \nu_1 \pi_{1i}(t) + \nu_2 \pi_{2i}(t) \tag{3.13}$$

with the initial condition

$$\pi_{ji}(0) = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases} \tag{3.14}$$

Solving the above equations (3.12) and (3.13) by Laplace transform we have

$$
\begin{aligned}
\pi_{11}^*(s) &= [6\beta_3^2+\beta_2\nu_1]/2D_2(\beta_2+\nu_1+\nu_2)\\
\pi_{12}^*(s) &= [3\nu_1+3\nu_2+\beta_2]/2D_2\\
\pi_{13}^*(s) &= \beta_2/2D_2\\
\pi_{21}^*(s) &= \nu_1\beta_2/2D_2(\beta_2+\nu_1+\nu_2)\\
\pi_{22}^*(s) &= 3(3\nu_1+3\nu_2+\beta_2)/2D_2\\
\pi_{23}^*(s) &= 3\beta_2/2D_2\\
\pi_{31}^*(s) &= \nu_1(3\beta_2+2\nu_1+2\nu_2)/2D_2(\beta_2+\nu_1+\nu_2)\\
\pi_{32}^*(s) &= (\nu_1+3\nu_2)/2D_2\\
\pi_{33}^*(s) &= 3(3\beta_2+2\nu_1+2\nu_2)/2D_2
\end{aligned}
\tag{3.15}
$$

where $s=2(\beta_2+\nu_1+\nu_2)$ and $D_2=9(\beta_2+\nu_1+\nu_2)^2+\beta_2\nu_1$. Using these π_{ji}^* we find that $h_{sty}(t)$ is now given by

$$
\begin{aligned}
h_{sty}(t) = \frac{\eta^2\beta_2^2\beta_3^2}{4D_1^2} &+ \{\eta^2\beta_2\beta_3 \exp[-2(\beta_2+\nu_1+\nu_2)t]\\
&\times[D_1D_2\lambda^2+2D_1(\lambda+\beta_2+\nu_1+\nu_2)(\beta_2+\nu_1+\nu_2)\\
&\times(\nu_1\beta_2+3\nu_2\beta_2+9\nu_3\beta_2+6\nu_1\nu_3+6\nu_3\nu_2)\\
&+2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)\\
&\times(-5\nu_1^3\beta_2+\nu_1\nu_2^2\beta_2-5\beta_2^3\nu_1-7\nu_1^2\nu_2\beta_2-5\nu_1\nu_2\beta_2^2\\
&-11\nu_1^2\beta_2^2+3\nu_2^3\beta_2+3\nu_2\beta_2^3+6\nu_2^2\beta_2^2-35\beta_2^2\nu_1\nu_3\\
&-37\beta_2\nu_1^2\nu_3-70\nu_1\nu_2\nu_3\beta_2-30\nu_2\nu_3\beta_2^2\\
&-33\beta_2\nu_3\nu_2^2-36\nu_1^2\nu_2\nu_3-36\nu_1\nu_2^2\nu_3-9\beta_2^3\nu_3\\
&-12\nu_1^3\nu_3-12\nu_2^3\nu_3)]\}/[8D_1^2D_2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)]\\
&+[\eta^2\beta_2\beta_3\exp[-(\beta_2+\nu_1+\nu_2)t]\cos(\tfrac{1}{2}\Gamma t)(\beta_2+\nu_1+\nu_2)\\
&\times(4\nu_1\nu_3\beta_2+3\nu_2\nu_3\beta_2+3\nu_1^2\nu_3+6\nu_1\nu_2\nu_3\\
&+3\nu_2^2\nu_3-\nu_1^2\beta_2-4\nu_1\nu_2\beta_2-\beta_2^2\nu_1-3\nu_2^2\beta_2-3\nu_2\beta_2^2)]/(D_1^2D_2)\\
&+\{2\beta_2^2\beta_3\nu_1(\beta_2+\nu_1+\nu_2)[\nu_3(3\beta_2+2\nu_1+2\nu_2)+\nu_2\beta_2-3(\nu_1+\nu_2+\beta_2)^2]\\
&\times\exp[-(\beta_2+\nu_1+\nu_2)t\sin(\tfrac{1}{2}\Gamma t)\}(\Gamma D_1^2D_2).
\end{aligned}
\tag{3.16}
$$

If at this stage we set $\nu_1=\nu_2=\lambda$, $\nu_3=11\lambda/3$, $\beta_2=2\sqrt{2}\lambda/3$ we find that the coefficient of $\sin\Gamma t/2$ in (3.16) vanishes and $\mathscr{B}$, the measure of bunching which can also be defined as $h_{st}(0)/h_{sty}(\infty)$ (see, for example, Srinivasan 1988a) now takes the value 1.8743; besides, the expression on the right-hand side considerably simplifies and resembles the stationary value of the product density of degree two of the detection process of photons resulting from an amplitude mixture of coherent and chaotic beams of light. We will return to this aspect in the next section and show that the coefficients do really correspond to those characteristics of the mixture, provided the parameters are chosen appropriately. In the next section we will pursue this point and identify the intensity correlation.

4. A model of cavity radiation and detection

Cavity radiation and its detection has been studied from many points of view. Shimoda *et al* (1957) essentially viewed the radiation as an assembly of photons arising from a population process with birth, death and immigration. Scully and Lamb (1966, 1967) used a fully quantum mechanical approach and dealt with the elements of the density matrix. By appropriate coarse-graining and elimination of the atomic coordinates they established the validity of the population point process approach. Shepherd (1981) and Shepherd and Jakeman (1987) clarified many of the features of the population approach and established a correspondence between the population and field parameters. These models have been further improvised by Srinivasan (1988a) who incorporated memory effects and dealt with the non-Markov evolution of the population leading to spectra of the resulting radiation that are observed experimentally in different contexts. There is a third approach due to Haken (1981) who deals with the equation satisfied by the field operators. Haken starts with the Heisenberg equation satisfied by the field operators of the system consisting of an atom, field and an appropriate bath. The atomic and bath operators are then eliminated and the resulting equation is identified to be of Langevin type when suitable approximations are made.

While population models can be justified on the lines of argument provided earlier by Srinivasan and Vasudevan (1987), it may be worthwhile to establish that non-Markov evolution essentially arises from the elimination of atomic and bath variables. We shall demonstrate this for a simple system consisting of the field and a bath in which case we have (see Haken 1981)

$$H_0 = h\omega_0 b^+ b \tag{4.1}$$

$$H_B = \sum_\omega h\omega B_\omega^+ B_\omega \qquad \text{(bath)} \tag{4.2}$$

$$H_I = h \sum_\omega (g_\omega b^+ B_\omega + g_\omega^+ B_\omega^+ b). \tag{4.3}$$

The corresponding Heisenberg equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} b^+ = \mathrm{i}\omega_0 b^+ + \mathrm{i} \sum_\omega g_\omega^+ B_\omega^+ \tag{4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} B_\omega^+ = \mathrm{i}\omega B_\omega^+ + \mathrm{i} g_\omega b^+. \tag{4.5}$$

The above equation leads to

$$B_\omega^+(t) = \mathrm{i} \int_0^t b^+(\tau) g_\omega \exp[\mathrm{i}\omega(t-\tau)]\,\mathrm{d}\tau + B_\omega^+(0)\exp(\mathrm{i}\omega t) \tag{4.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} b^+(t) = \mathrm{i}\omega_0 b^+ - \int_0^t b^+(\tau) \sum_\omega |g_\omega|^2 \exp[\mathrm{i}\omega(t-\tau)]\,\mathrm{d}\tau + \mathrm{i}\sum_\omega g_\omega^+ B_\omega^+(0)\exp(\mathrm{i}\omega t). \tag{4.7}$$

The above equation is the starting point in a series of approximations employed by Haken who arrived at a Langevin equation for the field operator b or b^+. We note that the last term in (4.7) can be identified to be the inhomogeneous forcing term in the Langevin equation while the second term under suitable approximation can be identified to be the contribution due to dissipation. If, however, no such drastic approximations are made then the second term is essentially a memory dependent term. Herein lies the origin of the non-Markovian nature of the evolution of the population of photons. The phase approach used by Srinivasan implicitly makes a particular choice of g_ω. Viewed from this angle the different types of non-Markov evolution essentially arise from a different choice of g_ω. Now we can interpret the results obtained in sections 2 and 3 in the context of the population of photons in a cavity. The emigration process is nothing other than the process of detection. The final results of section 3 relate to the first two moments of the steady-state photon population. The quantity $A_1(\infty)$ denotes the mean population (intensity) and $B_1(\infty)$ gives the second factorial moment. The quantity $\mathscr{B}$ itself is the measure of bunching. It is clear from the model that there is bunching and the model should correspond to thermal photons. To identify the radiation we look at the detection process. From the results of section 3 it follows that the average rate of detection can be identified as $f_1(\ )$ so that

$$f_1(\ ) = \eta\beta_2\beta_3/2D_1. \tag{4.8}$$

On the other hand, $h_{sty}(t)(t>0)$ denotes the stationary intensity correlation at time point t separated by t. An examination of the expression $h_{sty}(t)$ shows that the intensity correlation consists of sine and cosine terms. It is possible by a choice of the parameters of the population evolution process to eliminate the sine term so that the intensity correlation looks like the one corresponding to the radiation obtained by the amplitude mixing of thermal and coherent light. To emphasise this point further we have chosen the parameters in such a way that the intensities of the coherent and chaotic parts can be identified. If we now choose the parameters

$$\nu_3 = (6\nu_1 + 5\nu_2)/3 \tag{4.9}$$

and

$$\beta_2 = [(\nu_1+\nu_2)(3\nu_1+\nu_2)]^{1/2}/3 \tag{4.10}$$

we have

$$\begin{aligned}
h_{sty}(t) = {} & \frac{\eta^2\beta_2^2\beta_3^2}{4D_1^2} + \{\eta^2\beta_2\beta_3 \exp[-2(\beta_2+\nu_1+\nu_2)t] \\
& \times\{D_1D_2\lambda^2 + 2D_1(\lambda+\beta_2+\nu_1+\nu_2)(\beta_2+\nu_1+\nu_2) \\
& \times(\nu_1\beta_2 + 3\nu_2\beta_2 + 9\nu_3\beta_2 + 6\nu_1\nu_3 + 6\nu_3\nu_2) \\
& + 2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2) \\
& \times(\nu_1\nu_2^2\beta_2 - 5\nu_1^3\beta_2 - 5\beta_2^3\nu_1 - 7\nu_1^2\nu_2\beta_2 - 5\nu_1\nu_2\beta_2^2 \\
& - 11\nu_1^2\beta_2^2 + 3\nu_2^3\beta_2 + 3\nu_2\beta_2^3 + 6\nu_2^2\beta_2^2 - 35\beta_2^2\nu_1\nu_3 - 37\beta_2\nu_1^2\nu_3 \\
& - 70\nu_1\nu_2\nu_3\beta_2 - 30\nu_2\nu_3\beta_2^2 - 33\beta_2\nu_3\nu_2^2 - 36\nu_1^2\nu_2\nu_3 \\
& - 36\nu_1\nu_2^2\nu_3 - 9\beta_2^3\nu_3 - 12\nu_1^3\nu_3 - 12\nu_2^3\nu_3)]\}/ \\
& \times[8D_1^2D_2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)] \\
& + [\eta^2\beta_2\beta_3 \exp[-(\beta_2+\nu_1+\nu_2)t]\cos(\tfrac{1}{2}\Gamma t)(\beta_2+\nu_1+\nu_2) \\
& \times(4\nu_1\nu_3\beta_2 + 3\nu_2\nu_3\beta_2 + 3\nu_1^2\nu_3 + 6\nu_1\nu_2\nu_3 + 3\nu_2^2\nu_3 \\
& - \nu_1^2\beta_2 - 4\nu_1\nu_2\beta_2 - \beta_2^2\nu_1 - 3\nu_2^2\beta_2 - 3\nu_2\beta_2^2)]/(D_1^2D_2).
\end{aligned} \tag{4.11}$$

The intensity correlation corresponding to the radiation resulting from amplitude mixing of coherent and chaotic light has been extensively discussed in the literature (Jakeman and Pike 1969). If we adapt the results of Jakeman and Pike and express it in the standard notation of population point process theory we can identify $h_{sty}(t)$ to be

$$h_{sty}(t)=(\bar{I}_{ch}+\bar{I}_{co})^2+\bar{I}_{ch}^2\exp(-2\xi|t|)+2\bar{I}_{ch}\bar{I}_{co}\exp(-\xi|t|)\cos\theta t \qquad (4.12)$$

where $\bar{I}_{ch}$ and $\bar{I}_{co}$ are given by

$$\begin{aligned}\bar{I}_{ch}=(\{\eta^2\beta_2\beta_3[&(D_1D_2\lambda^2+2D_1(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)\\&\times(\nu_1\beta_2+3\nu_2\beta_2+9\nu_3\beta_2+6\nu_1\nu_3+6\nu_3\nu_2)+2(\beta_2+\nu_1+\nu_2)\\&\times(\lambda+\beta_2+\nu_1+\nu_2)(\nu_1\nu_2^2\beta_2-5\nu_1^3\beta_2-5\beta_2^3\nu_1-7\nu_1^2\nu_2\beta_2\\&-5\nu_1\nu_2\beta_2^2-11\nu_1^2\beta_2^2+3\nu_2^3\beta_2+3\nu_2\beta_2^3+6\nu_2^2\beta_2^2-35\beta_2^2\nu_1\nu_3-37\beta_2\nu_1^2\nu_3\\&-70\nu_1\nu_2\nu_3\beta_2-30\nu_2\nu_3\beta_2^2-33\beta_2\nu_3\nu_2^2-36\nu_1^2\nu_2\nu_3-36\nu_1\nu_2^2\nu_3-9\beta_2^3\nu_3\\&-12\nu_1^3\nu_3-12\nu_2^3\nu_3)]\}/[8D_1^2D_2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)])^{1/2}\end{aligned} \qquad (4.13)$$

$$\begin{aligned}\bar{I}_{co}=([2\eta^2\beta_2\beta_3&(\beta_2+\nu_1+\nu_2)^3(\lambda+\beta_2+\nu_1+\nu_2)\\&\times(4\nu_1\nu_3\beta_2+3\nu_2\nu_3\beta_2+3\nu_1^2\nu_3+6\nu_1\nu_2\nu_3+3\nu_2^2\nu_3\\&-\nu_1^2\beta_2-4\nu_1\nu_2\beta_2-\beta_2^2\nu_1-3\nu_2^2\beta_2-3\nu_2\beta_2^2)^2]/\\&\times\{D_1^2D_2[D_1D_2\lambda^2+2D_1(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)\\&\times(\nu_1\beta_2+3\nu_2\beta_2+9\nu_3\beta\\&\times(\lambda+\beta_2+\nu_1+\nu_2)(\nu_1\nu_2^2\beta_2-5\nu_1^3\beta_2-5\nu_1\beta_2^3-7\nu_1^2\nu_2\beta_2-5\nu_1\nu_2\beta_2^2\\&-11\nu_1^2\beta_2^2+3\nu_2^3\beta_2+3\nu_2\beta_2^3+6\nu_2^2\beta_2^2-35\beta_2^2\nu_1\nu_3-37\beta_2\nu_1^2\nu_3-70\nu_1\nu_2\nu_3\beta_2\\&-30\nu_2\nu_3\beta_2^2-33\beta_2\nu_3\nu_2^2-36\nu_1^2\nu_2\nu_3-36\nu_1\nu_2^2\nu_3-9\beta_2^3\nu_3\\&-12\nu_1^3\nu_3-12\nu_2^3\nu_3)]\})^{1/2}.\end{aligned} \qquad (4.14)$$

For instance, if the parameters satisfy the following two choices:

$$\begin{aligned}D_1D_2\lambda^2&+2D_1(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)(\nu_1\beta_2+3\nu_2\beta_2+9\nu_3\beta_2+6\nu_1\nu_3+6\nu_3\nu_2)\\&+2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)\\&\times(\nu_1\nu_2^2\beta_2-5\nu_1^3\beta_2-5\beta_2^3\nu_1-7\nu_1^2\nu_2\beta_2-5\nu_1\nu_2\beta_2^2\\&-11\nu_1^2\beta_2^2+3\nu_2^3\beta_2+3\nu_2\beta_2^3+6\nu_2^2\beta_2^2-35\beta_2^2\nu_1\nu_3-37\beta_2\nu_1^2\nu_3\\&-70\nu_1\nu_2\nu_3\beta_2-30\nu_2\nu_3\beta_2^2-33\beta_2\nu_3\nu_2^2-36\nu_1^2\nu_2\nu_3\\&-36\nu_1\nu_2^2\nu_3-9\beta_2^3\nu_3-12\nu_1^3\nu_3-12\nu_2^3\nu_3)\\&=[\beta_2\beta_3D_2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)]/18 \qquad \text{Choice 1}\end{aligned} \qquad (4.15)$$

$$\begin{aligned}D_1D_2\lambda^2&+2D_1(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)(\nu_1\beta_2+3\nu_2\beta_2+9\nu_3\beta_2+6\nu_1\nu_3+6\nu_3\nu_2)\\&+2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)\\&\times(\nu_1\nu_2^2\beta_2-5\nu_1^3\beta_2-5\beta_2^3\nu_1-7\nu_1^2\nu_2\beta_2-5\nu_1\nu_2\beta_2^2\\&-11\nu_1^2\beta_2^2+3\nu_2^3\beta_2+3\nu_2\beta_2^3+6\nu_2^2\beta_2^2-35\beta_2^2\nu_1\nu_3-37\beta_2\nu_1^2\nu_3\\&-70\nu_1\nu_2\nu_3\beta_2-30\nu_2\nu_3\beta_2^2-33\beta_2\nu_3\nu_2^2-36\nu_1^2\nu_2\nu_3\\&-36\nu_1\nu_2^2\nu_3-9\beta_2^3\nu_3-12\nu_1^3\nu_3-12\nu_2^3\nu_3)\\&=[\beta_2\beta_3D_2(\beta_2+\nu_1+\nu_2)(\lambda+\beta_2+\nu_1+\nu_2)]/2 \qquad \text{Choice 2}\end{aligned} \qquad (4.16)$$

we have

$$\bar{I}_{\mathrm{ch}}=\eta\beta_2\beta_3/12D_1 \qquad \text{for Choice 1} \tag{4.17}$$

$$\bar{I}_{\mathrm{ch}}=\eta\beta_2\beta_3/4D_1 \qquad \text{for Choice 2} \tag{4.18}$$

and

$$\bar{I}_{\mathrm{co}}=[6\eta(\beta_2+\nu_1+\nu_2)(4\nu_1\nu_3\beta_2+3\nu_2\nu_3\beta_2+3\nu_1^2\nu_3+6\nu_1\nu_2\nu_3+3\nu_2^2\nu_3-\nu_1^2\beta_2$$
$$-4\nu_1\nu_2\beta_2-\beta_2^2\nu_1-3\nu_2^2\beta_2-3\nu_2\beta_2^2)]/D_1D_2 \qquad \text{for Choice 1} \tag{4.19}$$

$$\bar{I}_{\mathrm{co}}=[2\eta(\beta_2+\nu_1+\nu_2)(4\beta_2\nu_1\nu_3+3\beta_2\nu_2\nu_3+3\nu_3(\nu_1+\nu_2)^2-\beta^2(\nu_1+3\nu_2)$$
$$-\beta(\nu_1^2+3\nu_2^2)-4\beta_2\nu_1\nu_2)]/D_1D_2 \qquad \text{for Choice 2.} \tag{4.20}$$

If we further impose the conditions

$$(4\nu_1\nu_3\beta_2+3\nu_2\nu_3\beta_2+3\nu_1^2\nu_3+6\nu_1\nu_2\nu_3+3\nu_2^2\nu_3-\nu_1^2\beta_2-4\nu_1\nu_2\beta_2$$
$$-\beta_2^2\nu_1-3\nu_2^2\beta_2-3\nu_2\beta_2^2$$
$$=5\beta_2\beta_3D_2/72(\beta_2+\nu_1+\nu_2) \qquad \text{Choice 1} \tag{4.21}$$

$$(4\beta_2\nu_1\nu_3+3\beta_2\nu_2\nu_3+3\nu_3(\nu_1+\nu_2)^2-\beta_2^2(\nu_1+3\nu_2)-\beta_2(\nu_1^2+3\nu_2^2)-4\beta_2\nu_1\nu_2)$$
$$=\beta_2\beta_3D_2/8(\beta_2+\nu_1+\nu_2) \qquad \text{Choice 2} \tag{4.22}$$

we get

$$\bar{I}_{\mathrm{co}}=5\eta\beta_2\beta_3/12D_1 \qquad \text{for Choice 1} \tag{4.23}$$

$$\bar{I}_{\mathrm{co}}=\eta\beta_2\beta_3/4D_1 \qquad \text{for Choice 2.} \tag{4.24}$$

We have checked the feasibility of the above relations in the sense that physically meaningful values of the parameters do exist and satisfy the above constraints. Although the expressions (4.16), (4.21), (4.19) and (4.24) look very compact, the choice of the parameters is by no means trivial since constraints like (4.15) and (4.18) have to be imposed. This particular situation is due to the fact that we laid more emphasis on the feasibility of the choice. However we can proceed as follows and obtain a fairly tractable form for $\bar{I}_{\mathrm{ch}}$ and $\bar{I}_{\mathrm{co}}$. Instead of imposing the constraint (4.9) and (4.10) we first make the coefficient of $\sin \Gamma t/2$ to vanish by a direct choice of ν_3:

$$\nu_3=[3(\nu_1+\nu_2+\beta_2)^2-\nu_2\beta_2]/(3\beta_2+2\nu_1+2\nu_2). \tag{4.25}$$

We then further set $\nu_1=\nu_2=\beta_2=\nu$ (say). We note that β_1 has already been fixed in arriving at the expressions for the moments $A_1(t)$, $A_2(t)$ and $A_3(t)$ (see (2.21), (2.24) and (2.26)). Finally we choose $\lambda=k\nu$. With this choice of parameters $h_{st_y}(t)$ as given by (4.11) reduces to the form (4.12) provided k satisfies (4.15). On reduction it was found that k satisfies a second-degree equation, one of whose roots is 6.32. So with this choice of k and the usual understanding that $\eta=1$ we get $\bar{I}_{\mathrm{ch}}=4.949\,8182$ and

$$\bar{I}_{\mathrm{co}}=0.114\,5421.$$

The coherent component $\bar{I}_{\mathrm{co}}$ arises essentially due to the situation that the state of immigration makes a re-entry to state 3 at a rate equal to $\nu_3=26\nu/7$ and it is this that is responsible for Poisson-like emissions. It is also to be specially noted that, if ν_1 and ν_2 were to be equal to zero, then the resulting stream would be thermal in nature. An appropriate choice of ν_1 and ν_2 leads to the persistence of the coherent component.

The model we have discussed is the most general one when the centre frequencies of the coherent and chaotic beams are distinct. It is possible to discuss a simpler situation when the two frequencies coincide, in which case $\cos \Gamma t/2$ occurring in (4.11) becomes unity. This can be achieved by setting $\nu_1 = 0$. In fact, if we set $\nu_1 = 0$ in (2.12) and (2.13) phase 1 automatically gets eliminated since it is essentially a transient one. Thus it is sufficient if we consider a two-phase model of immigration. Arguments similar to those in section 2 lead to the following differential equations for the generating functions:

$$\partial g_i(z,t)/\partial t = -\rho g_i(z,t) + \lambda g_{i+1}(z,t) + \mu + \eta \qquad i = 1, 2, \dots, n-1 \tag{4.26}$$

$$\partial g_n(z,t)/\partial t = -(\rho+\alpha) g_n(z,t) + \alpha g_n(z,t) g_1(z,t) + \rho \tag{4.27}$$

with the initial condition

$$g_i(z,0) = z \qquad i = 1, 2, \dots, n \tag{4.28}$$

and

$$\partial G_1(z,t)/\partial t = -\alpha_1 G_1(z,t) + \alpha_1 G_2(z,t) \tag{4.29}$$

$$\partial G_2(z,t)/\partial t = -\alpha_2 G_2(z,t) + [\nu_1 G_1(z,t) + \nu_2 G_2(z,t)] g_1(z,t) \tag{4.30}$$

where $\nu_i = \alpha_2 p_i$, $i = 1, 2$, with the initial condition

$$G_i(z,0) = 1 \qquad i = 1, 2. \tag{4.31}$$

Explicit solutions for $G_i (i = 1, 2)$ are not possible but their moments can be obtained.

4.1. Moments of the population

Using the same notation as in section 2 we have the following differential equations for A_j and $B_j (j = 1, 2)$ as

$$(\mathrm{d}/\mathrm{d}t) A_1(t) = -\alpha_1 A_1(t) + \alpha_1 A_2(t) \tag{4.32}$$

$$(\mathrm{d}/\mathrm{d}t) A_2(t) = -\nu_1 A_2(t) + \nu_1 A_1(t) + (\nu_1 + \nu_2) a_1(t) \tag{4.33}$$

and

$$(\mathrm{d}/\mathrm{d}t) B_1(t) = -\alpha_1 B_1(t) + \alpha_1 B_2(t) \tag{4.34}$$

$$(\mathrm{d}/\mathrm{d}t) B_2(t) = -\nu_1 B_2(t) + \nu_1 B_1(t) + (\nu_1 + \nu_2) b_1(t) + 2L(t) \tag{4.35}$$

with

$$A_j(0) = B_j(0) = 0. \tag{4.36}$$

The Laplace transforms solution of the $A_j (j = 1, 2)$ are

$$A_1^*(s) = \alpha_1 \alpha_2 / s(s + \mu + \eta)(s + \alpha_1 + \nu_1) \tag{4.37}$$

$$A_2^*(s) = \alpha_2 (s + \alpha_1) / s(s + \mu + \eta)(s + \alpha_1 + \nu_1). \tag{4.38}$$

On inversion and making the choice $\mu + \eta = 2(\alpha_1 + \nu_1)$, we get

$$A_1(t) = \alpha_1 \alpha_2 / (\alpha_1 + \nu_1)^2 \{\tfrac{1}{2} - \exp[-(\alpha_1 + \nu_1)t] + \tfrac{1}{2} \exp[-2(\alpha_1 + \nu_1)t\} \tag{4.39}$$

$$A_2(t) = \alpha_2 / (\alpha_1 + \nu_1)^2 \{\tfrac{1}{2}\alpha_1 + \nu_1 \exp[-(\alpha_1 + \nu_1)t] - \tfrac{1}{2}(\alpha_1 + 2\nu_1) \exp[-2(\alpha_1 + \nu_1)t]. \tag{4.40}$$

Similarly the Laplace transform solution of the B_j are given by

$$B_1^*(s) = \alpha_1\alpha_2 b_1^*(s)/s(s+\alpha_1+\nu_1) + 2\alpha_1 L^*(s)/s(s+\alpha_1+\nu_1) \tag{4.41}$$

$$B_2^*(s) = \alpha_2(s+\alpha_1) b_1^*(s)/s(s+\alpha_1+\nu_1) + 2(s+\alpha_1) L^*(s)/s(s+\alpha_1+\nu_1) \tag{4.42}$$

where

$$\begin{aligned} L^*(s) &= \nu_1 A_1^*(s+2\alpha_1+2\nu_1) + \nu_2 A_2^*(s+2\alpha_1+2\nu_1) \\ b_1^*(0) &= \lambda^2/2(\alpha_1+\nu_1)(\rho^2-\lambda^2) \\ a_i(t) &= \exp[-2(\alpha_1+\nu_1)t] \qquad i = 1, 2, \ldots, n \end{aligned} \tag{4.43}$$

and the steady-state factorial moments are given by

$$B_1(\infty) = B_2(\infty) = \frac{\alpha_1\alpha_2\lambda^2}{2(\alpha_1+\nu_1)^2(\rho^2-\lambda^2)} + \frac{\alpha_1\alpha_2[\alpha_1\alpha_2 + 2\nu_2(\alpha_1+\nu_1)]}{12(\alpha_1+\nu_1)^4} \tag{4.44}$$

$$A_1(\infty) = A_2(\infty) = \alpha_1\alpha_2/2(\alpha_1+\nu_1)^2. \tag{4.45}$$

Following the arguments similar to those in section 3 we obtain the following results:

$$f_1(t) = \text{a constant} = \eta\alpha_1\alpha_2/2(\alpha_1+\nu_1)^2 \tag{4.46}$$

$$h_1^1(t) = \eta a_1(t) = \eta \exp[-2(\alpha_1+\nu_1)t] \tag{4.47}$$

$$h_{sty}(t) = \eta^2 B_1(\infty)\exp[-2(\alpha_1+\nu_1)t] + \frac{\eta^2\alpha_1}{(\alpha_1+\nu_1)} \sum_{i,i=1}^{2} \nu_j \pi_{ji}^*[2(\alpha_1+\nu_1)] A_i(t) \tag{4.48}$$

where $\nu_j = \alpha_2 p_j, j = 1, 2$. An explicit solution for $h_{sty}(t)$ is obtained once we determine π_{ji}^*. Proceeding as in section 3, we finally obtain

$$\begin{aligned} h_{sty}(t) = [\eta\alpha_1\alpha_2/2(\alpha_1+\nu_1)^2]^2 &+ \frac{\{\eta^2\alpha_1\alpha_2\nu_1(\nu_2-\alpha_1)\exp[-(\alpha_1+\nu_1)t\}}{3(\alpha_1+\nu_1)^4} + \frac{\eta^2\alpha_1\alpha_2}{2(\alpha_1+\nu_1)^3} \\ &\times \exp[-2(\alpha_1+\nu_1)t]\left(\frac{\lambda^2}{4(\lambda+\alpha_1+\nu_1)} + \frac{\nu_1(\alpha_1-\nu_2)}{3(\alpha_1+\nu_1)}\right). \end{aligned} \tag{4.49}$$

We now proceed to show that $h_{sty}(t)$ can be cast in the form

$$h_{sty}(t) = (\bar{I}_{ch} + \bar{I}_{co})^2 + \bar{I}_{ch}^2 \exp(-2\xi|t|) + 2\bar{I}_{ch}\bar{I}_{co}\exp(-\xi|t|) \tag{4.50}$$

from which we identify with $\eta = 1$:

$$\bar{I}_{ch} = \left(\frac{\alpha_1\alpha_2[3\lambda^2(\alpha_1+\nu_1) + 4\nu_1(\alpha_1-\nu_2)(\lambda+\alpha_1+\nu_1)]}{24(\alpha_1+\nu_1)^4(\alpha_1+\lambda+\nu_1)}\right)^{1/2} \tag{4.51}$$

$$\bar{I}_{co} = \left(\frac{\alpha_1\alpha_2[\nu_1(\nu_2-\alpha_1)]^2\{[6(\alpha_1+\nu_1+\lambda)]\}}{9(\alpha_1+\nu_1)^4[3\lambda^2(\alpha_1+\nu_1) + 4\nu_1(\alpha_1-\nu_2)(\lambda+\alpha_1+\nu_1)]}\right)^{1/2}. \tag{4.52}$$

4.2. *Choice I*

If the parameters satisfy the following condition:

$$3\lambda^2(\alpha_1+\nu_1) + 4\nu_1(\alpha_1-\nu_2)(\lambda+\alpha_1+\nu_1) = \alpha_1\alpha_2(\lambda+\alpha_1+\nu_1)/6 \tag{4.53}$$

we have

$$\bar{I}_{ch} = \frac{\eta\alpha_1\alpha_2}{12(\alpha_1+\nu_1)^2} \tag{4.54}$$

and

$$\bar{I}_{co} = \frac{2\eta\nu_1(\nu_2 - \alpha_1)}{(\alpha_1 + \nu_1)^2}. \tag{4.55}$$

Further, if we impose the additional condition:

$$\nu_1(\nu_2 - \alpha_1) = \tfrac{5}{24}\,\alpha_1\alpha_2 \tag{4.56}$$

we can substitute for ν_2 to obtain a simplified expression for $\bar{I}_{co}$:

$$\bar{I}_{co} = \frac{5\eta\alpha_1\alpha_2}{12(\alpha_1 + \nu_1)^2}. \tag{4.57}$$

4.3. Choice II

If the parameters satisfy the condition

$$3\lambda^2(\alpha_1 + \nu_1) + 4\nu_1(\alpha_1 - \nu_2)(\lambda + \alpha_1 + \nu_1) = 3\alpha_1\alpha_2(\lambda + \alpha_1 + \nu_1)/2 \tag{4.58}$$

we have

$$\bar{I}_{ch} = \frac{\eta\alpha_1\alpha_2}{4(\alpha_1 + \nu_1)^2} \tag{4.59}$$

and

$$\bar{I}_{co} = \frac{2\eta\nu_1(\nu_2 - \alpha_1)}{3(\alpha_1 + \nu_1)^2}. \tag{4.60}$$

Further, if we impose the additional condition:

$$\nu_1(\nu_2 - \alpha_1) = 3\alpha_1\alpha_2/8 \tag{4.61}$$

we can substitute for ν_2 to obtain a simplified expression for $\bar{I}_{co}$:

$$\bar{I}_{co} = \frac{\eta\alpha_1\alpha_2}{4(\alpha_1 + \nu_1)^2}. \tag{4.62}$$

The above relations are feasible, in the sense that physically meaningful values of the parameters do exist satisfying the above conditions. We can proceed on lines very similar to those leading to (4.11)-(4.24) and establish the feasibility for general values of the parameters. However, by taking the following specific choice of parameters: $\alpha_1 = \alpha$, $\nu_1 = m\alpha$, $\lambda = k\alpha$ we find that (4.56) leads to

$$\nu_2 = 29m\alpha/(24m - 5)$$

$$\alpha_2 = [24m(m+1)\alpha]/(24m - 5)$$

and the parameter k gets determined by the constraint (4.53). Thus $\bar{I}_{ch}$ and $\bar{I}_{co}$ are given by

$$\bar{I}_{ch} = 2m/[(m+1)(24m - 5)]$$

$$\bar{I}_{co} = 10m/[(m+1)(24m - 5)].$$

The constraint (4.56) really fixes the ratio of $\bar{I}_{ch}$ to $\bar{I}_{co}$. In fact we can make this ratio arbitrary by dropping this constraint which was introduced mainly to simplify the expressions which are rather unwieldy. Thus we can satisfactorily conclude that a special case of the model leads to a heterodyne statistics with respect to the distribution function obtained.

5. Multiplicity distribution in high-energy collision

We finally bring out the relevance of the population models to the problem of determination of multiplicity distribution in high-energy collisions and, in particular, in colliding-beam experiments. The problem of multiplicity distribution has been approached from many angles; however, we restrict our attention to the class of models that are inspired by thermal light model, particularly by Carruthers and Shih (1987) on the one hand and Giovannini and Van Hove (1986) on the other (see also Srinivasan and Vasudevan 1988, Srinivasan 1988b, Giovannini 1979). There are many surveys now available on this topic (see, for example, Sarcevic 1987, Hwa 1988). To get a proper orientation we first discuss the relevance of the Shepherd (1981) model of cavity radiation and identify the population parameters. In the Carruthers and Shih (1987) model the equations are generally written for the probability mass function of the gluons. If we now use the notation $g(z, t)$ and $G(z, t)$ in the place of $g_i(z, t)(i = 1, 2, \ldots, n)$ and $G_j(z, t)(j = 1, 2, 3)$ in section 2 the equations which can now be interpreted to be the corresponding generating function of the number of gluons take the simple form

$$g'(z, t) = -(\lambda + \mu + \eta)g(z, t) + \lambda[g(z, t)]^2 + \mu + \eta \tag{5.1}$$

$$G'(z, t) = -\nu G(z, t) + \nu G(z, t)g(z, t). \tag{5.2}$$

We can effect further simplification by setting $\eta = 0$ since we are not considering the detection problem. In the Carruthers and Shih (1987) model the population parameters λ, μ and ν are given by the following choice:

$$\lambda = \lambda_0 \bar{n}$$

$$\nu = \lambda_0 \bar{n} k$$

$$\mu = \lambda_0 \bar{n} + k.$$

This particular choice is eminently reasonable since the main idea is to arrive at the gluon distribution which will ultimately represent the particle multiplicity distribution. The parameter $\bar{n}$ is chosen so as to provide agreement with the mean multiplicity of the particle distribution. We will not go into the detailed argument justifying the above choice since it has already been discussed in the literature (Sarcevic 1987, Carruthers and Shih 1987). The parameter k can be identified to be the number of clans introduced by Giovannini and Van Hove who have visualised the quark effect through the media of clans, each of which gives rise to a Poisson sequence of gluons which in turn generate a cascade. However the population model as put forward by Shepherd (1981) and summarised through (5.1) and (5.2) brings out the dynamics of the hard collision process; while (5.1) describes the hard process evolution through gluon multiplication, (5.2) describes the contribution from the gluon due to quark bremsstrahlung since the parameter k can be interpreted to be the intensity of the Poisson process of gluon emission by quarks during the process of hard collision. It is, of course, well known that such a model describes essentially a population with thermal characteristics. However it has been found that the particle multiplicity distribution also has characteristics similar to those of the stream obtained by amplitude mixing of coherent and chaotic beams of light (see, for example, Fowler *et al* 1988). It is in this context that the population model discussed in sections 2–4 becomes relevant to the description of particle multiplicity distributions. In particular, the properties relating to rapidity scaling of multiplicity distribution discussed by Fowler *et al* (1988) easily follow from the multiphase model of section 2. It is to be noted that by a proper choice of parameters the ratio of the strength of the chaotic to coherent field can be chosen to be equal to unity leading to forward-backward symmetry in rapidity distribution of the particles. It is also to be noted that the multiphase model can also lead to antibunching of the population. Thus the population model enables us to arrive at a dynamical interpretation of the particle multiplicity distributions that are generally derived from other considerations.

6. Summary and conclusion

We have analysed in this paper an age-dependent model of population growth. Age dependence itself is brought out by an evolution through phases whose durations form a family of independent non-negative random variables. This enables us to model the system by a simple differential equation. The moment structure of the population enable us to make useful inferences regarding the correlation properties of the population. Admittedly the population model was mainly inspired by cavity radiation and detection. In particular, an explicit expression is provided for the second-order intensity correlation which has the structure of that for the amplitude mixing of coherent and chaotic light. We also establish the relevance of this model to the thermal models that have been proposed for the study of multiplicity distribution of particles produced in high-energy collision. Such an identification enables us to throw new light on the dynamical interpretation of clan and thermal models. For instance, the forward-backward symmetry in the rapidity distribution of the particles essentially arises from the multiphase evolution of the immigration process. Thus we can infer that, in order to account for the experimentally observed forward-backward symmetry in rapidity distribution, a simple clan model involving Poisson emission of gluons as postulated by Giovannini and Van Hove may not be sufficient. Apparently the emissions are a little delayed and the delay can be accommodated within the framework of the phase model. The phase model by itself is viable and it is possible to include features like antibunching in which case it may be possible to accommodate the production of baryons as well.

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